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Entangled states, Lorentz transformations and spin precession in magnetic fields

A Chakrabarti

Centre de Physique Théorique, Ecole Polytechnique, 91128 Palaiseau Cedex, France

E-mail: chakra@cpht.polytechnique.fr

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Abstract
Two positive mass, spin-$\frac{1}{2}$ particles created in an entangled state are studied in the presence of a constant magnetic field inducing distinct precessions, depending on the respective momenta, of the two spins. The charge and anomalous magnetic moment of each particle are taken into account. Consequences for entanglement and, more generally, on correlations, are derived. We start, however, with a compact derivation of the effects of Lorentz transformations on such entangled states, although that has been studied by several authors. Our formalism displays conveniently the analogies and the differences between the two cases. Moreover, combining the two, one obtains the case of constant, orthogonal electric and magnetic fields. More general perspectives are evoked in the concluding remarks.

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1. Introduction

In studying entangled quantum states the particles involved are usually assumed to move freely outward from their point of production up to the apparatus, say Stern–Gerlach, where their polarizations are measured. We refer to the review of GHSZ [1] where a full, lucid discussion can be found with ample references to original sources. Two recent sources exploring the implications of non-locality and reality provide further insight and, again, ample references [2, 3].

Here we explore the consequences on the correlations observed of accelerations (Lorentz transformations) and of external constant magnetic fields. Combining these two we obtain also the effects of a 'crossed' electric field, meaning fields $(\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{B}})$, constant and orthogonal, satisfying

$$\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{B}} = 0, \qquad |\overrightarrow{\mathbf{E}}| < |\overrightarrow{\mathbf{B}}|. \tag{1.1}$$

We consider only massive ($m > 0$) spin-$\frac{1}{2}$ particle pairs with spin projections $\left(\pm\frac{1}{2}\right)$. There are several recent studies of entanglement in the context of Lorentz transformations [4–8]. We summarize however certain features (section 3) implementing our parametrization of Wigner rotations (appendix A). This turns out to be particularly helpful for our study of entangled states in a magnetic field (section 4).

How does the spin react to the Lorentz transformations and magnetic fields? The answer is through 'Wigner rotations'. A review [9] presents systematically our relevant results, referring to our previous papers (which will not be cited here separately). A comparative study, citing a very large number of sources, is provided by a recent review [10]. Wigner's construction of unitary representations of the Poincaré group leads to irreducible ones labeled by two invariants, one with continuous spectrum and the other with a discrete, integer or half-integer one. The first one is the mass (m) and the second one spin (s). (The zero-mass, continuous spin case does not seem to be realized in nature.) This is a profound result. The purely group theoretic construction leads directly to these two fundamental physical properties of a particle.

How do spins, thus obtained, behave? Wigner's construction again furnishes the answer in a canonical fashion for all spins. One thus arrives at Wigner rotations for spins (appendix A). Under pure rotations the spin and the momentum turn about the same axis through the same angle. Under pure Lorentz transformations, for $m > 0$, the spin turns again about the same axis but through a smaller angle than the momentum. There is a mass-dependent 'lag-effect'. (For $m = 0$, the spin catches up. The helicity remains constant.) The precession of polarization in a magnetic field, the full Thomas equation, can be obtained by starting with the time derivative of Wigner rotations [9]. In the following sections the consequences of Lorentz transformations and precessions for entangled states are studied systematically, implementing such an approach.

2. Recapitulation of basic features

Here, before even coming to Lorentz transformations, we summarize some well-known fundamental facts concerning two massive ($m > 0$), spin-$\frac{1}{2}$ particles in an entangled state. Rather than citing the famous original sources we refer conveniently to the appendices A, B, C, ... of [1]. Let

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2) \tag{2.1}$$

denote the state of total spin zero of two spin-$\frac{1}{2}$ particles, of 4-velocities $(u_0, \vec{\mathbf{u}})$, $(u_0, -\vec{\mathbf{u}})$ respectively in the frame of a particular observer (considered as the rest frame). The following facts are well known:

(i) The state is 'entangled'. Namely, the assumption that one can express it as

$$|\psi\rangle = |\omega\rangle_1|z\rangle_2, \tag{2.2}$$

where $|\omega\rangle_1$ and $|z\rangle_2$ are each a linear superposition of states in the space of states of particles 1 and 2 respectively, leads to a contradiction.

(ii) Rotating the axis of spin projections simultaneously for the two particles to any direction $\widehat{\mathbf{n}}$ (with $\widehat{\mathbf{n}}^2 = 1$) leaves the right-hand side of (2.1) invariant. This is an expression of a spherical symmetry of the spin zero state and holds as long as the axis of projection is the same ($\widehat{\mathbf{n}}_1 = \widehat{\mathbf{n}}_2 = \widehat{\mathbf{n}}$) for the two particles.

(iii) When the two axes are different ($\widehat{\mathbf{n}}_1 \neq \widehat{\mathbf{n}}_2$), taking (appendix B of [1]) $\widehat{\mathbf{n}}_2$ as the polar axis and $\widehat{\mathbf{n}}_1$ with polar and azimuthal angles $(\theta, 0)$ respectively one obtains

$$\begin{aligned}|\psi\rangle = \frac{1}{\sqrt{2}}\bigg(&-\sin\left(\frac{\theta}{2}\right)|\widehat{\mathbf{n}}_1,+\rangle_1|\widehat{\mathbf{n}}_2,+\rangle_2 + \cos\left(\frac{\theta}{2}\right)|\widehat{\mathbf{n}}_1,+\rangle_1|\widehat{\mathbf{n}}_2,-\rangle_2 \\ &- \cos\left(\frac{\theta}{2}\right)|\widehat{\mathbf{n}}_1,-\rangle_1|\widehat{\mathbf{n}}_2,+\rangle_2 - \sin\left(\frac{\theta}{2}\right)|\widehat{\mathbf{n}}_1,-\rangle_1|\widehat{\mathbf{n}}_2,-\rangle_2\bigg).\end{aligned} \tag{2.3}$$

The amplitudes for the joint outcomes are respectively ($\pm$ denoting the projections detected)

$$(P_{++}, P_{+-}, P_{-+}, P_{--}) = \frac{1}{2}\left(\sin^2\left(\frac{\theta}{2}\right), \cos^2\left(\frac{\theta}{2}\right), \cos^2\left(\frac{\theta}{2}\right), \sin^2\left(\frac{\theta}{2}\right)\right) \tag{2.4}$$

consistent with $P_{++} + P_{+-} + P_{-+} + P_{--} = 1$. Hence the expectation value of the product of the measurement outcomes is now

$$E(\widehat{\mathbf{n}}_1, \widehat{\mathbf{n}}_2) = P_{++} + P_{--} - P_{-+} - P_{+-} = \sin^2\left(\frac{\theta}{2}\right) - \cos^2\left(\frac{\theta}{2}\right) = -\cos\theta = -(\widehat{\mathbf{n}}_1 \cdot \widehat{\mathbf{n}}_2). \tag{2.5}$$

For $\widehat{\mathbf{n}}_1 = \widehat{\mathbf{n}}_2 = \widehat{\mathbf{n}}$ one obtains the perfect correlation ($\theta = 0$) as

$$E = -1. \tag{2.6}$$

(This is also called 'anti-correlation' since for $\theta = 0$ one obtains $(+, -)$ or $(-, +)$ projections only.)

(iv) For measurements involving three directions, $(\widehat{\mathbf{n}}_1, \widehat{\mathbf{n}}_2, \widehat{\mathbf{n}}_3)$ say, Bell's inequality can be expressed in our notation as

$$|E(\widehat{\mathbf{n}}_1, \widehat{\mathbf{n}}_2) - E(\widehat{\mathbf{n}}_1, \widehat{\mathbf{n}}_3)| - E(\widehat{\mathbf{n}}_2, \widehat{\mathbf{n}}_3) \leqslant 1. \tag{2.7}$$

Bell derived this (in 1964) in the context of the argument of Einstein, Podolski and Rosen (EPR) involving locality, reality and completeness. See, for example section 2 of [1] and corresponding references. It is a famous fact that quantum mechanical predictions can violate Bell's inequality. In particular, implementing (2.5), representing such predictions, the lhs of (2.7) becomes

$$|-(\widehat{\mathbf{n}}_1 \cdot \widehat{\mathbf{n}}_2) + (\widehat{\mathbf{n}}_1 \cdot \widehat{\mathbf{n}}_3)| + (\widehat{\mathbf{n}}_2 \cdot \widehat{\mathbf{n}}_3). \tag{2.8}$$

Choosing, for example, $(\widehat{\mathbf{n}}_1, \widehat{\mathbf{n}}_2, \widehat{\mathbf{n}}_3)$ in the xy-plane with azimuthal angles $\left(0, \frac{\pi}{3}, \frac{2\pi}{3}\right)$ respectively one obtains

$$|-(\widehat{\mathbf{n}}_1 \cdot \widehat{\mathbf{n}}_2) + (\widehat{\mathbf{n}}_1 \cdot \widehat{\mathbf{n}}_3)| + (\widehat{\mathbf{n}}_2 \cdot \widehat{\mathbf{n}}_3) = \tfrac{3}{2} > 1. \tag{2.9}$$

3. Lorentz transformations and correlations

Now we come to Lorentz transformations. Here, as in section 4, we restrict our considerations to eigenstates of equal and opposite initial momenta for the two particles. Density matrices are not introduced, as compared to some relevant recent sources already cited. The transitions between the results of sections 2, 3, 4 respectively are thus displayed conveniently and clearly. Consider an observer for whom the first frame (section 2) is related through a pure Lorentz transformation corresponding to a 4-velocity u''. We now implement the result of appendix A. To start with the 4-velocities of particle 1 and particle 2 (of (2.2)) are

$$(u_0, \vec{\mathbf{u}}) \qquad \text{and} \qquad (u_0, -\vec{\mathbf{u}}), \tag{3.1}$$

respectively. Using (A.1), (A.2) define the corresponding

$$a_1 = (1 + u_0)(1 + u_0'')(1 + u_0 u_0'' + \vec{\mathbf{u}} \cdot \vec{\mathbf{u}}''), \qquad b_1 = 1 + u_0 + u_0'' + u_0 u_0'' + \vec{\mathbf{u}} \cdot \vec{\mathbf{u}}'' \tag{3.2}$$

and

$$a_2 = (1 + u_0)(1 + u_0'')(1 + u_0 u_0'' - \vec{\mathbf{u}} \cdot \vec{\mathbf{u}}''), \qquad b_2 = 1 + u_0 + u_0'' + u_0 u_0'' - \vec{\mathbf{u}} \cdot \vec{\mathbf{u}}''. \tag{3.3}$$

The Wigner rotations of the spins (for $\overrightarrow{\mathbf{u}} \times \overrightarrow{\mathbf{u}}'' \neq 0$) will be respectively about the axes

$$\widehat{\mathbf{k}}_{(1)} = -\widehat{\mathbf{k}}_{(2)} = \widehat{\mathbf{k}} = \frac{\overrightarrow{\mathbf{u}} \times \overrightarrow{\mathbf{u}}''}{|\overrightarrow{\mathbf{u}} \times \overrightarrow{\mathbf{u}}''|} \tag{3.4}$$

through angles (δ_1, δ_2) where

$$\cos\left(\frac{\delta_1}{2}\right) = \frac{b_1}{\sqrt{2a_1}}, \qquad \cos\left(\frac{\delta_2}{2}\right) = \frac{b_2}{\sqrt{2a_2}}. \tag{3.5}$$

The spin states transform as

$$\begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix}_{(1)} \longrightarrow \mathrm{e}^{\mathbf{i}\delta_1 \widehat{k}\cdot\frac{\overrightarrow{\sigma}}{2}} \begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix}_{(1)}, \tag{3.6}$$

$$\begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix}_{(2)} \longrightarrow \mathrm{e}^{-\mathbf{i}\delta_2 \widehat{k}\cdot\frac{\overrightarrow{\sigma}}{2}} \begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix}_{(2)}. \tag{3.7}$$

(Here $\overrightarrow{\sigma}$ denote the Pauli matrices.) With direction cosines of $\widehat{\mathbf{k}}$ denoted as (k_1, k_2, k_3) satisfying

$$k_1^2 + k_2^2 + k_3^2 = \widehat{\mathbf{k}}^2 = 1 \tag{3.8}$$

one obtains for the entangled state (2.1)

$$\begin{aligned} \frac{1}{\sqrt{2}}(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2) \longrightarrow \frac{1}{\sqrt{2}}\Bigg(& \cos\frac{(\delta_1+\delta_2)}{2}(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2) \\ & + \mathbf{i}\sin\frac{(\delta_1+\delta_2)}{2}[k_3(|+\rangle_1|-\rangle_2 + |-\rangle_1|+\rangle_2) \\ & - (k_1 + \mathbf{i}k_2)|+\rangle_1|+\rangle_2 + (k_1 - \mathbf{i}k_2)|-\rangle_1|-\rangle_2]\Bigg). \end{aligned} \tag{3.9}$$

(For a rotation, one just sets $\delta_2 = -\delta_1$. A rotation $-\delta_1$ about $-\widehat{\mathbf{k}}$ is one of δ_1 about $\widehat{\mathbf{k}}$. One thus recovers the rotational invariance of the lhs as mentioned in section 2, subsection 2.) Fundamental consequences: we state them directly to start with:

(1) *Entanglement is frame independent*. Lorentz transformations do not 'disentangle' an entangled state. It remains entangled in all frames.

(2) *Violation of Bell's inequality is frame-dependent.* For suitable choices of parameters a violation of the inequality for the first observer can be absent for the second after a Lorentz transformation.

We now demonstrate statement (1) directly. Then we extract the consequences of a Lorentz transformation concerning correlation. This will lead to (2) along with other results.

Assume that the rhs of (3.9) can be expressed (as in (2.2)) in a non-entangled form as

$$(x_1|+\rangle_1 + y_1|-\rangle_1)(x_2|+\rangle_2 + y_2|-\rangle_2). \tag{3.10}$$

Comparing with (3.9) one obtains

$$\begin{aligned} x_1x_2 &= -\mathbf{i}(k_1 + \mathbf{i}k_2)\sin\frac{(\delta_1+\delta_2)}{2}, \\ y_1y_2 &= \mathbf{i}(k_1 - \mathbf{i}k_2)\sin\frac{(\delta_1+\delta_2)}{2}, \\ x_1y_2 &= \cos\frac{(\delta_1+\delta_2)}{2} + \mathbf{i}k_3\sin\frac{(\delta_1+\delta_2)}{2}, \\ x_2y_1 &= -\cos\frac{(\delta_1+\delta_2)}{2} + \mathbf{i}k_3\sin\frac{(\delta_1+\delta_2)}{2}. \end{aligned} \tag{3.11}$$

This implies

$$x_1x_2y_1y_2 = \left(k_1^2+k_2^2\right)\sin^2\frac{(\delta_1+\delta_2)}{2} = -\cos^2\frac{(\delta_1+\delta_2)}{2} - k_3^2\sin^2\frac{(\delta_1+\delta_2)}{2} \tag{3.12}$$

or, since (3.8) holds,

$$\cos^2\frac{(\delta_1+\delta_2)}{2} + \sin^2\frac{(\delta_1+\delta_2)}{2} = 0. \tag{3.13}$$

But, from our construction, (δ_1, δ_2) are real angles (see (3.5)) and hence

$$\cos^2\frac{(\delta_1+\delta_2)}{2} + \sin^2\frac{(\delta_1+\delta_2)}{2} = 1. \tag{3.14}$$

Thus one arrives at a contradiction. Hence statement (1) holds.

Let us now consider correlations as observed in the frame of the second observer. To simplify notations set, in (3.9),

$$\cos\frac{(\delta_1+\delta_2)}{2} \equiv c, \qquad \sin\frac{(\delta_1+\delta_2)}{2} \equiv s \tag{3.15}$$

and introduce

$$\cos\frac{\theta'}{2} \equiv c', \qquad \sin\frac{\theta'}{2} \equiv s' \tag{3.16}$$

for the new polar angle of the spin projections of particle 2 with the corresponding states

$$|+\rangle_2 = c'|+\rangle_2' + s'|-\rangle_2', \qquad |-\rangle_2 = -s'|+\rangle_2' + c'|-\rangle_2'. \tag{3.17}$$

The rhs of (3.9) is then

$$\tfrac{1}{\sqrt{2}}(\alpha_1|+\rangle_1|+\rangle_2' + \alpha_2|+\rangle_1|-\rangle_2' + \beta_1|-\rangle_1|+\rangle_2' + \beta_2|-\rangle_1|-\rangle_2'), \tag{3.18}$$

where

$$\begin{aligned} \alpha_1 &= (k_2sc' - cs') - \mathbf{i}s(k_1c' + k_3s'), \qquad & \beta_2 &= (k_2sc' - cs') + \mathbf{i}s(k_1c' + k_3s'), \\ \alpha_2 &= (k_2ss' + cc') - \mathbf{i}s(k_1s' - k_3c'), \qquad & \beta_1 &= -(k_2ss' + cc') - \mathbf{i}s(k_1s' - k_3c'). \end{aligned} \tag{3.19}$$

The amplitude for the spin projections (compare (2.4)) is now

$$\begin{aligned} P_{++} = P_{--} &= \tfrac{1}{2}((k_2sc' - cs')^2 + s^2(k_1c' + k_3s')^2), \\ P_{+-} = P_{-+} &= \tfrac{1}{2}((k_2ss' + cc')^2 + s^2(k_3c' - k_1s')^2). \end{aligned} \tag{3.20}$$

As a check one notes (using $c^2 + s^2 = c'^2 + s'^2 = k_1^2 + k_2^2 + k_3^2 = 1$)

$$P_{++} + P_{--} + P_{+-} + P_{-+} = 1. \tag{3.21}$$

The correlation is now (compare (2.5))

$$\begin{aligned} E &= P_{++} + P_{--} - P_{+-} - P_{-+} \\ &= (k_2s(c' + s') + c(c' - s'))(k_2s(c' - s') - c(c' + s')) \\ &\quad + s^2(k_1(c' - s') + k_3(c' + s'))(k_1(c' + s') - k_3(c' - s')). \end{aligned} \tag{3.22}$$

This varies with the parameters in a relatively complicated fashion. Corresponding to particular choices of the axis of the Wigner rotation ($\hat{\mathbf{k}} \approx \vec{\mathbf{u}} \times \vec{\mathbf{u}}$") one obtains simple particular cases of interest.

Case 1.

$$k_1 = k_2 = 0, \qquad k_3 = \pm 1, \tag{3.23}$$

$$E = -(c'^2 - s'^2) = -\cos\theta'. \tag{3.24}$$

Case 2.

$$k_3 = k_1 = 0, \qquad k_2 = \pm 1, \tag{3.25}$$

$$E = -((c^2 - s^2)(c'^2 - s'^2) \pm (2cs)(2c's')) = -\cos(\delta_1 + \delta_2 \mp \theta'). \tag{3.26}$$

For, respectively, $\delta_1 + \delta_2 = \pm\theta'$ the (anti)correlation becomes perfect for the second observer.

Case 3.

$$k_2 = k_3 = 0, \qquad k_1 = \pm 1, \tag{3.27}$$

$$E = -(c^2 - s^2)(c'^2 - s'^2) = -\cos(\delta_1 + \delta_2)\cos\theta'. \tag{3.28}$$

This last case suffices to illustrate the possibility presented as statement (2) concerning the frame dependence of the inequality. Varying θ' corresponding to $(\widehat{\mathbf{n}}_1, \widehat{\mathbf{n}}_2, \widehat{\mathbf{n}}_3)$ involved in (2.7), (2.8) and (2.9) (namely our θ') the rhs of (2.9) becomes (due to the extra factor in (3.25)), for the second observer

$$\tfrac{3}{2}\cos(\delta_1 + \delta_2). \tag{3.29}$$

For

$$\cos(\delta_1 + \delta_2) < \tfrac{2}{3} \tag{3.30}$$

Bell's inequality is satisfied in the frame of the second observer while it is violated in that of the first. We have thus established its frame dependence.

4. The entangled state in a constant magnetic field

The equations for the precession of canonical polarization operators are presented in appendix B. We recapitulate briefly the basic equations and final results. More details can be found in appendix B. Let $(\overrightarrow{\mathbf{B}}, \overrightarrow{\mathbf{v}}, \overrightarrow{\Sigma})$ be respectively the constant, homogeneous magnetic field, the velocity and the polarization. With unit vectors $(\widehat{\mathbf{B}}, \widehat{\mathbf{v}})$,

$$\overrightarrow{\mathbf{B}} = B\widehat{\mathbf{B}}, \qquad \overrightarrow{\mathbf{v}} = v\widehat{\mathbf{v}}, \qquad \gamma = (1 - v^2)^{-1/2}, \qquad \alpha = (g-2)/2 \tag{4.1}$$

and

$$\overrightarrow{\varpi} = \frac{eB}{m\gamma}\widehat{\mathbf{B}}, \qquad \overrightarrow{\Omega} = \frac{\alpha e B}{m\gamma}(\gamma\widehat{\mathbf{B}} - (\gamma - 1)(\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})\widehat{\mathbf{v}}) \tag{4.2}$$

the equations are

$$\frac{\mathrm{d}\overrightarrow{\mathbf{v}}}{\mathrm{d}t} = -\overrightarrow{\varpi} \times \overrightarrow{\mathbf{v}}, \qquad \frac{\mathrm{d}\overrightarrow{\Sigma}}{\mathrm{d}t} = -(\overrightarrow{\varpi} + \overrightarrow{\Omega}) \times \overrightarrow{\Sigma}. \tag{4.3}$$

Apart from $(v, \widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})$ one has constants

$$\omega^2 = \left(\frac{eB}{m\gamma}\right)^2, \qquad \Omega^2 = \left(\frac{\alpha e B}{m\gamma}\right)^2 (\gamma^2 - (\gamma^2 - 1)(\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})^2), \tag{4.4}$$

where with unit vectors $(\widehat{\omega}, \widehat{\Omega})\overrightarrow{\varpi} = \omega\widehat{\omega}$, $\overrightarrow{\Omega} = \Omega\widehat{\Omega}$. For

$$0 \leqslant (\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})^2 < 1 \tag{4.5}$$

one starts with ((0) denoting initial value at $t = 0$) ortho-normalized fixed axes

$$\widehat{\mathbf{B}}, \qquad \frac{(\widehat{\mathbf{B}} \times \widehat{\mathbf{v}})_{(0)}}{\sqrt{1 - (\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})^2}}, \qquad \frac{\widehat{\mathbf{B}} \times (\widehat{\mathbf{B}} \times \widehat{\mathbf{v}})_{(0)}}{\sqrt{1 - (\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})^2}}. \tag{4.6}$$

Then one defines ortho-normalized axes rotating about $\widehat{\mathbf{B}}$ as (using values at time t)

$$\widehat{\mathbf{B}}, \quad \frac{(\widehat{\mathbf{B}} \times \widehat{\mathbf{v}})}{\sqrt{1-(\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})^2}}, \qquad \frac{\widehat{\mathbf{B}} \times (\widehat{\mathbf{B}} \times \widehat{\mathbf{v}})}{\sqrt{1-(\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})^2}}. \tag{4.7}$$

The projections of $\overrightarrow{\Omega}$ are

$$\begin{aligned}
\Omega_1 &= \widehat{\mathbf{B}} \cdot \overrightarrow{\Omega} = \frac{\alpha e B}{m\gamma}(\gamma - (\gamma-1)(\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})^2),\\
\Omega_2 &= \frac{(\widehat{\mathbf{B}} \times \widehat{\mathbf{v}})\cdot\overrightarrow{\Omega}}{\sqrt{1-(\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})^2}} = 0,\\
\Omega_3 &= \frac{(\widehat{\mathbf{B}} \times (\widehat{\mathbf{B}} \times \widehat{\mathbf{v}}))\cdot\overrightarrow{\Omega}}{\sqrt{1-(\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})^2}} = \frac{\alpha e B}{m\gamma}(\gamma-1)(\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})\sqrt{1-(\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})^2},
\end{aligned} \tag{4.8}$$

consistent with $\Omega_1^2 + \Omega_3^2 = \Omega^2$. The solutions (presented in detail in appendix B) combine successive rotations of $\overrightarrow{\Sigma}$ through

(i) an angle (ωt) about the axis $-\widehat{\omega}(= -\widehat{\mathbf{B}})$,
(ii) an angle (Ωt) about the axis $-\widehat{\Omega}$.

From (B.8) it follows that in passing from particle 1 with initial velocity $\overrightarrow{\mathbf{v}}_{(0)}$ to particle 2 with $-\overrightarrow{\mathbf{v}}_{(0)}$ is equivalent to

$$\omega t \longrightarrow \omega t + \pi. \tag{4.9}$$

Thus the transformation matrices for the spin-$\frac{1}{2}$ states $(|+\rangle, |-\rangle)$ are respectively

$$\mathrm{e}^{-\mathbf{i}\frac{\Omega t}{2}(\widehat{\Omega}\cdot\overrightarrow{\sigma})}\,\mathrm{e}^{-\mathbf{i}\frac{\omega t}{2}(\widehat{\mathbf{B}}\cdot\overrightarrow{\sigma})} \tag{4.10}$$

for particle 1, and

$$\mathrm{e}^{-\mathbf{i}\frac{\Omega t}{2}(\widehat{\Omega}\cdot\overrightarrow{\sigma})}\,\mathrm{e}^{-\mathbf{i}\frac{(\omega t+\pi)}{2}(\widehat{\mathbf{B}}\cdot\overrightarrow{\sigma})} \tag{4.11}$$

for particle 2. Here

$$\widehat{\Omega}\cdot\overrightarrow{\sigma} = \frac{1}{\Omega}(\Omega_1\sigma_1 + \Omega_3\sigma_3). \tag{4.12}$$

For $(\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}}) = 0$ the two axes coincide and one has simply the matrices

$$\mathrm{e}^{-\mathbf{i}(\frac{\Omega+\omega}{2})t(\widehat{\mathbf{B}}\cdot\overrightarrow{\sigma})} \tag{4.13}$$

and

$$\mathrm{e}^{-\mathbf{i}(\frac{(\Omega+\omega)t+\pi}{2})(\widehat{\mathbf{B}}\cdot\overrightarrow{\sigma})} \tag{4.14}$$

for the particles 1 and 2, respectively. (N.B: the implementation of (4.9) has already taken care of the effect of the change of sign of $\overrightarrow{\mathbf{v}}_0$ on the axis of projection corresponding to Ω_3 through that on $(\widehat{\mathbf{B}}\times\widehat{\mathbf{v}})$. This is the content of (B.8). After this one should not include an additional inversion of sign of the factor $(\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})$ of Ω_3. This would amount to a double negative on one part and lead to inconsistent results—such as time-dependent precession of a spin zero state.)

Case 1: $(\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}} = 0)$. We start the study of correlations with the simple case corresponding to (4.13). We keep, however, general direction cosines of $\overrightarrow{\mathbf{B}}$, so that

$$\widehat{\mathbf{B}}\cdot\overrightarrow{\sigma} = (b_1\sigma_1 + b_2\sigma_2 + b_3\sigma_3), \tag{4.15}$$

where by definition,

$$b_1^2 + b_2^2 + b_3^2 = 1. \tag{4.16}$$

Set $c = \cos\frac{1}{2}(\Omega+\omega)t$, $s = \sin\frac{1}{2}(\Omega+\omega)t$. From (4.13), the time evaluations of particles 1 and 2 are respectively

$$\begin{pmatrix} |+\rangle_1 \\ |-\rangle_1 \end{pmatrix}_{(t)} = \begin{bmatrix} c - \mathbf{i}b_3 s & -\mathbf{i}(b_1+\mathbf{i}b_2)s \\ -\mathbf{i}(b_1+\mathbf{i}b_2)s & c+\mathbf{i}b_3 s \end{bmatrix} \begin{pmatrix} |+\rangle_1 \\ |-\rangle_1 \end{pmatrix}_{(0)}, \tag{4.17}$$

$$\begin{pmatrix} |+\rangle_2 \\ |-\rangle_2 \end{pmatrix}_{(t)} = \begin{bmatrix} -s - \mathbf{i}b_3 c & -\mathbf{i}(b_1-\mathbf{i}b_2)c \\ -\mathbf{i}(b_1+\mathbf{i}b_2)c & -s+\mathbf{i}b_3 c \end{bmatrix} \begin{pmatrix} |+\rangle_2 \\ |-\rangle_2 \end{pmatrix}_{(0)}. \tag{4.18}$$

Implementing $c^2+s^2=1$, $b_1^2+b_2^2+b_3^2=1$, one obtains

$$\begin{aligned}(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2)_{(t)} &= -\mathbf{i}(b_1+\mathbf{i}b_2)|+\rangle_1|+\rangle_2 + \mathbf{i}(b_1-\mathbf{i}b_2)|-\rangle_1|-\rangle_2 \\ &\quad + \mathbf{i}b_3(|+\rangle_1|-\rangle_2 + |-\rangle_1|+\rangle_2).\end{aligned} \tag{4.19}$$

All time dependence disappears (through $c^2+s^2=1$) in the spin-zero entangled state. This is consistent with the fact that a zero spin does not undergo precession. The structure on the right-hand side of (4.19) should be compared to that of (2.3). But it remains entangled. The demonstration is parallel to (3.10)–(3.14) (for change of frame). Assume that one can express (4.19) as

$$(x_1|+\rangle_1 + y_1|-\rangle_1)(x_2|+\rangle_2 + y_2|-\rangle_2). \tag{4.20}$$

Then one must have

$$x_1x_2 = -\mathbf{i}(b_1+\mathbf{i}b_2), \qquad y_1y_2 = \mathbf{i}(b_1-\mathbf{i}b_2), \qquad x_1y_2 = \mathbf{i}b_3, \qquad x_2y_1 = \mathbf{i}b_3 \tag{4.21}$$

and hence $x_1x_2y_1y_2 = b_1^2+b_2^2 = -b_3^2$ or

$$b_1^2+b_2^2+b_3^2 = 0. \tag{4.22}$$

This contradicts (4.16). Hence (4.19) represents an entangled state.

Probabilities and correlations. We again proceed as in section 3 (see (3.16)–(3.25)). Rotate the states of particle 2 as

$$|+\rangle_2 \longrightarrow c'|+\rangle_2 + s'|-\rangle_2, \qquad |-\rangle_2 \longrightarrow -s'|+\rangle_2 + c'|-\rangle_2, \tag{4.23}$$

where $c' = \cos(\theta'/2)$, $s' = \sin(\theta'/2)$. Including a normalization factor $\frac{1}{\sqrt{2}}$, the rhs of (4.19) is now

$$\begin{aligned}&\frac{\mathbf{i}}{\sqrt{2}}[(-(b_1+\mathrm{i}b_2)c' - b_3s')|+\rangle_1|+\rangle_2 + ((b_1-\mathrm{i}b_2)c' + b_3s')|-\rangle_1|-\rangle_2 \\ &\quad - ((b_1+\mathrm{i}b_2)s' - b_3c')|+\rangle_1|-\rangle_2 - ((b_1-\mathrm{i}b_2)s' - b_3c')|-\rangle_1|+\rangle_2].\end{aligned} \tag{4.24}$$

The corresponding probabilities are

$$\begin{aligned} P_{++} = P_{--} &= \tfrac{1}{2}\big((b_1^2+b_2^2)c'^2 + b_3^2s'^2 + 2b_1b_3c's'\big), \\ P_{+-} = P_{-+} &= \tfrac{1}{2}\big((b_1^2+b_2^2)s'^2 + b_3^2c'^2 - 2b_1b_3c's'\big)\end{aligned} \tag{4.25}$$

satisfying

$$P_{++} + P_{--} + P_{+-} + P_{-+} = \big(b_1^2+b_2^2+b_3^2\big)(s'^2+c'^2) = 1. \tag{4.26}$$

The correlation is

$$\begin{aligned} E = P_{++} + P_{--} - P_{+-} - P_{-+} &= \big(b_1^2+b_2^2-b_3^2\big)(c'^2-s'^2) + (2b_1b_3)(2c's') \\ &= \big(b_1^2+b_2^2-b_3^2\big)\cos\theta' + (2b_1b_3)\sin\theta' \\ &= \cos\theta' - 2b_3(b_3\cos\theta' - b_1\sin\theta').\end{aligned} \tag{4.27}$$

As in section 3 one can now consider consequences for different choices of the parameters (compare with (3.22)–(3.25)). Thus, for example, setting

$$b_2 = 0, \qquad b_3 = \cos(\varphi/2), \qquad b_1 = \sin(\varphi/2), \tag{4.28}$$

$$E = -\cos(\varphi + \theta'). \tag{4.29}$$

For $b_3 = 0$,

$$E = \cos\theta'. \tag{4.30}$$

For $b_1 = 0, b_3 = \cos(\psi/2), b_2 = \sin(\psi/2)$

$$E = \left(1 - 2b_3^2\right)\cos\theta' = -\cos\psi . \cos\theta'. \tag{4.31}$$

Consequences analogous to those of section 3 can be observed. Thus (compare (3.25)–(3.27)), for the same strength of the homogeneous magnetic field (B) rotating it (changing ψ, say, in (4.31)) one can satisfy or violate Bell's inequality, (for the same $(\widehat{\mathbf{n}}_1, \widehat{\mathbf{n}}_2, \widehat{\mathbf{n}}_3)$ as in the passage from (2.9) to (3.26)).

Case 2: $(0 < (\widehat{\mathbf{B}} \cdot \overrightarrow{\mathbf{v}})^2 < 1)$. We now have the general spinor matrices (4.10) and (4.11) for particles 1 and 2, respectively. Having illustrated the consequences of rotating the magnetic field $\overrightarrow{\mathbf{B}}$ (i.e., variations of (b_1, b_2, b_3) in the preceding subsection (for $\overrightarrow{\mathbf{B}} \cdot \overrightarrow{\mathbf{v}} = 0$) let us simplify the formalism here (for non-zero $\overrightarrow{\mathbf{B}} \cdot \overrightarrow{\mathbf{v}}$) by choosing axes such that (conserving (4.12))

$$\overrightarrow{\mathbf{B}} \cdot \overrightarrow{\sigma} = B\sigma_1, \qquad \overrightarrow{\Omega} \cdot \overrightarrow{\sigma} = \Omega_1\sigma_1 + \Omega_3\sigma_3. \tag{4.32}$$

For the limit $(\widehat{\mathbf{B}} \cdot \overrightarrow{\mathbf{v}}) = 0$ this corresponds to $b_1 = 1, b_2 = b_3 = 0$ leading in (4.19) to

$$(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2)_{(t)} = -\mathbf{i}(|+\rangle_1|+\rangle_2 - |-\rangle_1|-\rangle_2)_{(0)} \tag{4.33}$$

and the contradiction between (4.16) and (4.22) reduces to that between

$$b_1^2 = 1, \qquad b_1^2 = 0. \tag{4.34}$$

With notations

$$\frac{\Omega_1}{\Omega} = l_1, \qquad \frac{\Omega_3}{\Omega} = l_3, \qquad l_1^2 + l_3^2 = 1 \tag{4.35}$$

the spinor matrices for particles 1 and 2 respectively are now

$$M_1 = \mathrm{e}^{-\mathbf{i}\frac{\Omega t}{2}(l_1\sigma_1 + l_3\sigma_3)}\,\mathrm{e}^{-\mathbf{i}\frac{\omega t}{2}\sigma_1}, \qquad M_2 = \mathrm{e}^{-\mathbf{i}\frac{\Omega t}{2}(l_1\sigma_1 + l_3\sigma_3)}\,\mathrm{e}^{-\mathbf{i}\frac{(\omega t + \pi)}{2}\sigma_1}. \tag{4.36}$$

Set

$$\begin{aligned} \cos\left(\tfrac{1}{2}\omega t\right) &= c, \qquad & \sin\left(\tfrac{1}{2}\omega t\right) &= s, \\ \cos\left(\tfrac{1}{2}\Omega t\right) &= c', \qquad & \sin\left(\tfrac{1}{2}\Omega t\right) &= s' \end{aligned} \tag{4.37}$$

and

$$\alpha = (c'c - l_1 s' s) - \mathbf{i} l_3 s' c, \qquad \beta = (c's + l_1 s' c) - \mathbf{i} l_3 s' s. \tag{4.38}$$

Then

$$M_1 = \begin{bmatrix} \alpha & -\mathbf{i}\beta \\ -\mathbf{i}\beta^* & \alpha^* \end{bmatrix}, \qquad M_2 = \begin{bmatrix} -\beta & -\mathbf{i}\alpha \\ -\mathbf{i}\alpha^* & -\beta^* \end{bmatrix}. \tag{4.39}$$

Thus though (α, β) are complicated, (M_1, M_2) have simple structures in their terms. From (4.38) along with $c^2 + s^2 = 1 = c'^2 + s'^2$) one obtains

$$\alpha\alpha^* + \beta\beta^* = c'^2 + \left(l_1^2 + l_3^2\right)s'^2 = c'^2 + s'^2 = 1. \tag{4.40}$$

This implies the unitarity constraints

$$M_1^+ M_1 = M_2^+ M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.41}$$

Now

$$\begin{aligned}(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2)_{(t)} &= [-(\alpha|+\rangle_1 - \mathrm{i}\beta|-\rangle_1)(\mathrm{i}\alpha^*|+\rangle_2 + \beta^*|-\rangle_2) \\ &\quad + (-\mathrm{i}\beta^*|+\rangle_1 + \alpha^*|-\rangle_1)(\beta|+\rangle_2 + \mathrm{i}\alpha|-\rangle_2)]_{(0)} \\ &= -\mathbf{i}(\alpha\alpha^* + \beta\beta^*)(|+\rangle_1|+\rangle_2 - |-\rangle_1|-\rangle_2)_{(0)} \\ &\quad - (\alpha\beta^* + \alpha\beta^*)(|+\rangle_1|-\rangle_2)_{(0)} + (-\beta\alpha^* + \alpha^*\beta)(|-\rangle_1|+\rangle_2)_{(0)} \\ &= -\mathbf{i}(|+\rangle_1|+\rangle_2 - |-\rangle_1|-\rangle_2)_{(0)}.\end{aligned} \tag{4.42}$$

Thus not only again has time dependence disappeared for the spin-zero entangled state but again for non-zero $(\widehat{\mathbf{B}} \cdot \overrightarrow{\mathbf{v}})$ the relation (4.33) (for $(\widehat{\mathbf{B}} \cdot \overrightarrow{\mathbf{v}} = 0)$, $b_2 = b_3 = 0$) is reproduced. Now one can rotate the axes to give more general orientations to $(\widehat{\mathbf{B}}, \widehat{\Omega})$. The consequences can be deduced in a straightforward fashion. We will not go through the steps here.

Case 3: $(\widehat{\mathbf{B}} \cdot \overrightarrow{\mathbf{v}} = \pm 1)$. As discussed in appendix B ((B.25)–(B.26)), this case has to be treated separately. The normalization (4.6) is no longer well defined. But now v is constant. Thus this case can be treated quite simply as a separate one. We will not present such a discussion here.

5. Constant, orthogonal electric and magnetic fields

We only briefly indicate certain possibilities. Two cases have been studied in our previous papers and solutions have been obtained for the polarization. They are briefly presented in our review ([9]) with references to our original works.

Case 1: $(|\overrightarrow{\mathbf{E}}| < |\overrightarrow{\mathbf{B}}|)$. This is presented in section 3 of [9] (3.17–23). A Lorentz transformation corresponding to the 4-velocity

$$u'' = \frac{1}{\sqrt{1 - \frac{E^2}{B^2}}}\left(1, \frac{E}{B}\widehat{\mathbf{E}} \times \widehat{\mathbf{B}}\right), \tag{5.1}$$

where $(\widehat{\mathbf{E}}, \widehat{\mathbf{B}})$ are unit vectors gives transformed fields

$$\overrightarrow{\mathbf{E}}' = 0, \qquad \overrightarrow{\mathbf{B}}' = \left(1 - \frac{E^2}{B^2}\right)^{-1/2} \overrightarrow{\mathbf{B}}. \tag{5.2}$$

Thus combining the results presented here in sections 3 and 4 one can analyze this case fully.

Case 2. $(|\overrightarrow{\mathbf{E}}| = |\overrightarrow{\mathbf{B}}|)$ (See (4.13–14) of [9] and sources cited there.) The Dirac equation (with the Pauli term for anomalous magnetic moment and also electric dipole moment) has been solved for an external plane wave field. A particular limiting case corresponds to the present one. Complete solutions for polarization were obtained. The general plane wave will be studied elsewhere in the context of entanglement.

6. Remarks

We have studied the effects of accelerations (Lorentz transformations) and constant, homogeneous magnetic fields on the entangled state of two positive mass, spin-$\frac{1}{2}$ particles.

An additional constant electric field orthogonal to the magnetic one has also been briefly considered. We approach both aspects systematically via Wigner rotations of canonical spin, recapitulated in appendices A and B. Relevant basic features of entanglement (correlations and Bell's inequality) are recapitulated in section 2. (We consider both these domains as well known and refer to thorough review articles rather than to original sources.)

Considering frames of observers related through Lorentz transformations we have shown that entanglement is frame independent but the violation of Bell's inequality is frame dependent. (Compare [4–8].) We show that analogous features arise as the spin undergoes precession in a magnetic field. Correlations are formulated for the above cases. We note how, though the spins rotate in a magnetic field, the time dependence disappears for the entangled state, consistently with its total spin zero.

The effects of a Lorentz transformation are uniform in space time. A magnetic field introduces time dependence through spin rotations but is homogeneous in space.

What happens when particles created in an entangled state find themselves plunged into a space-time dependent external field? Among other possible consequences the spin correlations can depend on the location in space and time of the measurements carried out. Does the entanglement survive uniformly?

The Dirac equation (generalized to include terms corresponding to anomalous magnetic moment and even electric dipole moment) has been solved for a class of plane wave fields which includes plane-polarized ones. (See [9] and sources cited there.) Apart from special features arising from 4-component Dirac spinors involved, more fundamentally, space-time dependence is present.

Spinors have been studied in black hole metrics. A standard reference is section 10 of [11]. The Kerr–Schild formalism is elucidated in [12]. Entanglement beyond special relativity has been considered in [6]. In such contexts how close can one bring together the mysteries of entanglement and black holes? What happens when entangled particles start moving away along geodesics? We hope to understand such aspects better in the future.

To conclude let us emphasize some points making our motivations more explicit. We have displayed the consequences of Lorentz transformations and spin precession, as far as possible in a parallel fashion in sections 3 and 4. (Many results concerning Lorentz transformations can be found elsewhere. One has only to see the references cited, such as [6] and sources cited there. Here we present some results in section 3 implementing a formalism which leads, as directly as possible, to the next case.) There is however a fundamental difference which one should not forget. The magnetic field induces continuous variations (precessions) of the spins. Such a time dependence is entirely absent for Lorentz transformations. In (3.26) and (3.30) the consequences of the acceleration are encapsulated in the presence of the Wigner rotations given by two angles derived in (3.2), (3.3) and (3.5). There is no time dependence in these angles. This is to be compared to (4.37), (4.38) and (4.39) displaying the time dependences involved. One might have thought that here the entanglement would become time dependent in a periodic fashion varying between maxima and minima. Remarkably for the specific entangled state considered the time dependence cancels out in (4.42). As we have already noted above this is consistent with the zero total spin of our state. A zero spin cannot undergo precession. If the difference of the two terms on the left-hand side of (4.42) is replaced by, say, the sum (corresponding to projection zero of a total spin 1) quite evidently the time dependence no longer cancels. We would like to add that section 4 becomes possible due to the complete, explicit solutions obtained (in (B.20) along with (B.8)) for arbitrary relative orientations of the initial velocity, spin and the magnetic field.

We have next pointed out, without going into details, the consequences of crossed electric ($\vec{\mathbf{E}}$) and magnetic ($\vec{\mathbf{B}}$) fields in section 5. One motivation for this is that one can include the presence of such an electric field by combining the formalisms of sections 3 and 4, introducing thus a new physical background without additional complications. A more thorough exploration of the consequences of such a joint presence would be desirable.

Another interesting generalization of our formalism would be to consider three spinning particles. There one can compare the consequences concerning entanglement of two alternative coupling schemes:

(i) The standard two-step one, coupling via C.G. coefficients any two of the three particles (with three possible choices) and then the third with each irreducible component obtained in the first step.

(ii) An alternative single-step one which diagonalizes the mixed product of the three angular momentum operators leading to a simultaneous reduction with respect to the rotation and the permutation groups and thus to remarkable symmetry properties [13].

The consequent possible variations in the correlation functions (or the 'degree' of entanglement) obtained in these two schemes is an interesting aspect which we intend to explore elsewhere.

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Appendix A. Lorentz transformations and Wigner rotations of spin

The full treatment of Wigner rotations of polarization, the expectation values of the canonical spin operator, can be found in [9]. We present below briefly the essential results (based on equations (2.24–34) of [9]). Let u be the 4-velocity of a particle, of mass m and spin s, in a particular frame. A pure Lorentz transformation corresponding to a 4-velocity u'' will give one u', with

$$u_0' = u_0 u_0'' + \overrightarrow{u} \cdot \overrightarrow{u}''. \tag{A.1}$$

Define

$$a = (1 + u_0)(1 + u_0'')(1 + u_0'), \qquad b = 1 + u_0 + u_0'' + u_0'. \tag{A.2}$$

The momentum and the polarization both turn about the axis $\overrightarrow{u} \times \overrightarrow{u}''$, but through angles α and δ respectively where

$$\cos^2\frac{\delta}{2} = \frac{b^2}{2a}, \qquad \cos(\alpha - \delta) = \frac{u_0 u_0' - u_0''}{|\overrightarrow{\mathbf{u}}||\overrightarrow{\mathbf{u}}'|} = \frac{p_0 p_0' - m p_0''}{|\overrightarrow{\mathbf{p}}||\overrightarrow{\mathbf{p}}'|}. \tag{A.3}$$

This displays the 'lag' mentioned in the introduction. Evidently this lag vanishes as $m \to 0$. One can also write

$$\sin\delta = \frac{b}{a}|\overrightarrow{\mathbf{u}} \times \overrightarrow{\mathbf{u}}''|, \qquad \cos\delta = 1 - \frac{1}{a}|\overrightarrow{\mathbf{u}} \times \overrightarrow{\mathbf{u}}''|^2. \tag{A.4}$$

This displays explicitly the fact that for

$$\overrightarrow{\mathbf{u}} \times \overrightarrow{\mathbf{u}}'' = 0, \qquad \delta = 0. \tag{A.5}$$

Thus, in our canonical formulation, the polarization does not rotate under a pure Lorentz transformation parallel (or antiparallel) to the initial momentum. (Hence, the prescription of defining the polarization by passing to the rest frame is totally superfluous in our canonical formalism. One defines it directly in any frame, introducing canonical spin matrices.)

For a spin-$\frac{1}{2}$ particle a rotation δ about an axis $\widehat{\mathbf{k}}$ (say, with $\widehat{\mathbf{k}}^2 = 1$) transforms the basis states $|\pm\rangle$ as, with $\overrightarrow{\sigma}$ denoting Pauli matrices

$$e^{\mathbf{i}\frac{\delta}{2}\widehat{\mathbf{k}}\cdot\overrightarrow{\sigma}}\begin{pmatrix}|+\rangle\\|+\rangle\end{pmatrix} = \begin{pmatrix}\left(\cos\frac{\delta}{2} + \mathbf{i}\widehat{k}_3\sin\frac{\delta}{2}\right)|+\rangle + \mathbf{i}(\widehat{k}_1 - \mathbf{i}\widehat{k}_2)\sin\frac{\delta}{2}|-\rangle\\ \mathbf{i}(\widehat{k}_1 + \mathbf{i}\widehat{k}_2)\sin\frac{\delta}{2}|+\rangle + \left(\cos\frac{\delta}{2} - \mathbf{i}\widehat{k}_3\sin\frac{\delta}{2}\right)|-\rangle\end{pmatrix}. \tag{A.6}$$

Appendix B. Precession of polarization in a constant magnetic field

The Thomas equation for precession of polarization was derived in our previous papers (cited in [9]) starting with Wigner rotations of canonical spin. Solutions were presented. Here we present the solutions in a way compact and well adapted to our present goal. The key is the choice of axes displaying clearly the role of initial conditions, with study of correlations in view. Let $\overrightarrow{\mathbf{B}}$ be the constant magnetic field and $\overrightarrow{\mathbf{v}}$ the velocity of the positive m, spinning particle. Let $\overrightarrow{\mathbf{B}} = B\widehat{\mathbf{B}}$, $\overrightarrow{\mathbf{v}} = v\widehat{\mathbf{v}}$ ($\widehat{\mathbf{B}}^2 = \widehat{\mathbf{v}}^2 = 1$). (We use standard units with $c = 1$, $v < 1$.) Two constants of motion are v and $\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}}$. we start with the general case

$$0 \leqslant (\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})^2 < 1. \tag{B.1}$$

The limiting case

$$\widehat{\mathbf{B}}\times\widehat{\mathbf{v}} = 0, \qquad \widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}} = \pm 1 \tag{B.2}$$

is particularly simple and best treated separately. The case

$$\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}} = 0 \tag{B.3}$$

is also simple, but contained in (B.1). We choose, for (B.1), the ortho-normalized fixed axes

$$\widehat{\mathbf{B}}, \quad \frac{(\widehat{\mathbf{B}}\times\widehat{\mathbf{v}})_{(0)}}{\sqrt{1-(\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})^2}}, \quad \frac{\widehat{\mathbf{B}}\times(\widehat{\mathbf{B}}\times\widehat{\mathbf{v}})_{(0)}}{\sqrt{1-(\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})^2}}, \tag{B.4}$$

where (0) denotes the initial value at $t = 0$. The corresponding components of the polarization $\overrightarrow{\Sigma}$ are defined as

$$\Sigma_1 = \widehat{\mathbf{B}}\cdot\overrightarrow{\Sigma}, \qquad \Sigma_2 = \frac{(\widehat{\mathbf{B}}\times\widehat{\mathbf{v}})_{(0)}\cdot\overrightarrow{\Sigma}}{\sqrt{1-(\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})^2}}, \qquad \Sigma_3 = \frac{(\widehat{\mathbf{B}}\times(\widehat{\mathbf{B}}\times\widehat{\mathbf{v}})_{(0)})\cdot\overrightarrow{\Sigma}}{\sqrt{1-(\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})^2}}. \tag{B.5}$$

Next one defines the ortho-normalized rotating set of axes (with values at time t)

$$\widehat{\mathbf{B}}, \quad \frac{(\widehat{\mathbf{B}}\times\widehat{\mathbf{v}})}{\sqrt{1-(\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})^2}}, \quad \frac{\widehat{\mathbf{B}}\times(\widehat{\mathbf{B}}\times\widehat{\mathbf{v}})}{\sqrt{1-(\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})^2}}. \tag{B.6}$$

The rotation is also about $\widehat{\mathbf{B}}$, given by $\frac{d\widehat{\mathbf{v}}}{dt} = -\overrightarrow{\varpi}\times\widehat{\mathbf{v}}$ with

$$\overrightarrow{\varpi} = \frac{eB}{m\gamma}\widehat{\mathbf{B}} \qquad \gamma = (1-v^2)^{-1/2}. \tag{B.7}$$

One has (with $\overrightarrow{\varpi} = \omega\widehat{\omega}, \widehat{\omega}^2 = 1$)

$$\begin{aligned}(\widehat{\mathbf{B}}\times\widehat{\mathbf{v}}) &= (\widehat{\mathbf{B}}\times\widehat{\mathbf{v}})_{(0)}\cos\omega t - (\widehat{\mathbf{B}}\times(\widehat{\mathbf{B}}\times\widehat{\mathbf{v}}))_{(0)}\sin\omega t,\\ (\widehat{\mathbf{B}}\times(\widehat{\mathbf{B}}\times\widehat{\mathbf{v}})) &= (\widehat{\mathbf{B}}\times(\widehat{\mathbf{B}}\times\widehat{\mathbf{v}}))_{(0)}\cos\omega t + (\widehat{\mathbf{B}}\times\widehat{\mathbf{v}})_{(0)}\sin\omega t.\end{aligned} \tag{B.8}$$

One defines correspondingly the rotating components of $\overrightarrow{\Sigma}^{(r)}$ as

$$\Sigma_1^{(r)} = \Sigma_1, \qquad \Sigma_2^{(r)} = \Sigma_2\cos\omega t - \Sigma_3\sin\omega t, \qquad \Sigma_3^{(r)} = \Sigma_3\cos\omega t + \Sigma_2\sin\omega t. \tag{B.9}$$

The precession equation for $\vec{\Sigma}$ is

$$\frac{\mathrm{d}\vec{\Sigma}}{\mathrm{d}t} = -(\vec{\varpi} + \vec{\Omega}) \times \vec{\Sigma}, \tag{B.10}$$

where

$$\vec{\Omega} = \frac{\alpha e B}{m\gamma}(\gamma\widehat{\mathbf{B}} - (\gamma - 1)(\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})\widehat{\mathbf{v}}) \tag{B.11}$$

with $\alpha = (g-2)/2$ representing the effect of the anomalous magnetic moment. Carefully grouping the terms, after simplifications, one indeed obtains the expected result

$$\frac{\mathrm{d}\vec{\Sigma}^{(r)}}{\mathrm{d}t} = -\vec{\Omega} \times \vec{\Sigma}^{(r)}. \tag{B.12}$$

The effect of $\vec{\varpi}$ is absorbed in the rotating frame. One has just the supplementary rotation Ωt about $-\widehat{\Omega}$ (where $\vec{\Omega} = \Omega\widehat{\Omega}$, $\widehat{\Omega}^2 = 1$). In fact, since

$$(\widehat{\mathbf{B}} \times \widehat{\mathbf{v}}) \cdot \widehat{\Omega} = 0 \tag{B.13}$$

one obtains for $\vec{\Sigma}^{(r)}$

$$\frac{\mathrm{d}\Sigma_1^{(r)}}{\mathrm{d}t} = \Omega_3\Sigma_2^{(r)}, \qquad \frac{\mathrm{d}\Sigma_3^{(r)}}{\mathrm{d}t} = -\Omega_1\Sigma_2^{(r)}, \qquad \frac{\mathrm{d}\Sigma_2^{(r)}}{\mathrm{d}t} = \Omega_1\Sigma_3^{(r)} - \Omega_3\Sigma_1^{(r)} \tag{B.14}$$

with the constant coefficients

$$\begin{aligned}
\Omega_1 &= \widehat{\mathbf{B}}\cdot\vec{\Omega} = \frac{\alpha e B}{m\gamma}(\gamma - (\gamma-1)(\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})^2),\\
\Omega_2 &= \frac{(\widehat{\mathbf{B}}\times\widehat{\mathbf{v}})\cdot\vec{\Omega}}{\sqrt{1-(\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})^2}} = 0,\\
\Omega_3 &= \frac{(\vec{\mathbf{B}}\times(\widehat{\mathbf{B}}\times\widehat{\mathbf{v}}))\cdot\vec{\Omega}}{\sqrt{1-(\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})^2}} = \frac{\alpha e B}{m\gamma}(\gamma-1)(\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})\sqrt{1-(\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})^2}
\end{aligned} \tag{B.15}$$

with

$$\Omega_1^2 + \Omega_2^2 + \Omega_3^2 = \left(\frac{\alpha e B}{m}\right)^2 (1 - (\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})^2) = \Omega^2. \tag{B.16}$$

The explicit expressions for (Ω_1, Ω_3) are not particularly simple. But they are precisely what are needed to display the basic structure of the solutions. The linear set (B.14) with constant coefficient is easily solved. One obtains

$$\begin{aligned}
\Sigma_1^{(r)} &= \frac{\Omega_3}{\Omega}(a\cos\Omega t + b\sin\Omega t) + c\left(\frac{\Omega_1}{\Omega}\right),\\
\Sigma_3^{(r)} &= -\frac{\Omega_1}{\Omega}(a\cos\Omega t + b\sin\Omega t) + c\left(\frac{\Omega_3}{\Omega}\right),\\
\Sigma_2^{(r)} &= (b\cos\Omega t - a\sin\Omega t),
\end{aligned} \tag{B.17}$$

where (since $\vec{\Sigma}^{(r)}_{(0)} = \vec{\Sigma}_{(0)}$)

$$a = \frac{1}{\Omega}(\Omega_3\Sigma_{1(0)} - \Omega_1\Sigma_{3(0)}), \qquad c = \frac{1}{\Omega}(\Omega_1\Sigma_{1(0)} + \Omega_3\Sigma_{3(0)}), \qquad b = \Sigma_{2(0)} \tag{B.18}$$

and hence

$$|\vec{\Sigma}^{(r)}|^2 = a^2 + b^2 + c^2 = |\vec{\Sigma}^{(r)}_{(0)}|^2. \tag{B.19}$$

Finally one obtains the expected, standard, form

$$\vec{\Sigma}^{(r)} = (\cos\Omega t)\vec{\Sigma}_{(0)} + (1-\cos\Omega t)(\widehat{\Omega}\cdot\vec{\Sigma})_{(0)}\widehat{\Omega} - (\sin\Omega t)(\widehat{\Omega}\times\vec{\Sigma})_{(0)}. \tag{B.20}$$

This represents a rotation (Ωt) about the axis $(-\widehat{\Omega})$. Trivially inverting (B.8) one obtains the projections on fixed axes. Thus we have two successive rotations of

(i) angle (ωt) about the axis $(-\widehat{\omega})$,
(ii) angle (Ωt) about the axis $(-\widehat{\Omega})$.

The limit $(\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}}) = 0$. In this case the two axes coincide. One has just a rotation $(\Omega+\omega)t$ about $(-\widehat{\mathbf{B}})$ where

$$\vec{\Omega} = \alpha\gamma\vec{\omega}. \tag{B.21}$$

Now the axes (B.5) reduce to

$$(\widehat{\mathbf{B}}, \widehat{\mathbf{B}}\times\widehat{\mathbf{v}}, -\widehat{\mathbf{v}}) \tag{B.22}$$

and one has

$$\begin{aligned}(\widehat{\mathbf{B}}\times\widehat{\mathbf{v}}) &= (\widehat{\mathbf{B}}\times\widehat{\mathbf{v}})_{(0)}\cos\omega t + \widehat{\mathbf{v}}_{(0)}\sin\omega t,\\ \widehat{\mathbf{v}} &= \widehat{\mathbf{v}}_{(0)}\cos\omega t - (\widehat{\mathbf{B}}\times\widehat{\mathbf{v}})_{(0)}\sin\omega t.\end{aligned} \tag{B.23}$$

The special limit $(\widehat{\mathbf{B}}\times\widehat{\mathbf{v}}) = 0$. Now

$$1-(\widehat{\mathbf{B}}\cdot\widehat{\mathbf{v}})^2 = 0 \tag{B.24}$$

and one cannot normalize as in (B.5) and (B.6). But now $\widehat{v}$ remains constant and

$$\vec{\omega} = \frac{eB}{m\gamma}\widehat{\mathbf{B}}, \qquad \vec{\Omega} = \alpha\vec{\omega}. \tag{B.25}$$

(Note the difference of a factor γ for $\vec{\Omega}$, as compared to (B.21).) Choosing any pair of fixed ortho-normal axes in the plane orthogonal to $\vec{\mathbf{B}}$ one obtains quite simply the final results.

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